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COMPARISON OF SYNCHRONIZATION SPEED OF RING NETWORKS OF REACTION-DIFFUSION EQUATIONS OF FITZHUGH-NAGUMO TYPE WITH UNIDIRECTIONAL AND BIDIRECTIONAL COUPLING

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Abstract. The purpose of this work is to research the numerical results of the comparison of the synchronization speed of ring neural networks with unidirectional and bidirectional coupling. Each neuron is linearly coupled with the others and represented by a system of reaction-diffusion equations of the FitzHugh-Nagumo type. The result shows that the necessary coupling strength for the synchronization in two cases increases when the number of neurons increases. In other words, the greater the number of neurons in the ring networks is, the more difficult the synchronization occurs. Moreover, synchronizing the ring networks with bidirectional coupling is easier than the one with unidirectional coupling over the same given number of neurons.

Keywords: Bidirectional coupling, ring network, reaction-diffusion equations of FitzHugh-Nagumo type, synchronization, unidirectional coupling.

AMS Subject Classification: 34A34, 65L05.

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1 Introduction

Synchronization is an extremely important phenomenon in nature and nonlinear science, especially in the network of interconnected dynamical systems (Aziz-Alaoui, 2006; Keener et al., 2009; Murray, 2010). That means the systems will have the same behavior at the same time. Specifically, for a network of two systems, synchronization means that one system will copy the properties of the other from a certain time. Then, the network is said to be synchronous.

In the human brain, many cells connect to form a network of cells. A cellular network is a system of cells that are physiologically linked together. The exchange between them is mainly based on electrochemical processes. This paper numerically presents the synchronization in the ring networks of linearly interconnected cells. In these considered networks, each cell is described by a system of reaction-diffusion equations of FitzHugh-Nagumo type (FHN).

Hodgkin & Huxley (1952) proposed a four-dimensional mathematical model that could approximate the energizing properties of cell voltage Corson (2009). Based on this model, many simpler models have been published to describe the cell voltage dynamics. In 1962, FitzHugh and Nagumo published a new model named the FitzHugh-Nagumo model known as a simplified two-dimensional model from the famous system of Hodgkin-Huxley (Hodgkin & Huxley, 1952; Ambrosio et al., 2013; Izhikevich, 2007). Although the model is simpler, it has many remarkable

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analytical results and retains its properties and biological significance. This model is made up of two equations of two variables u and v. The first variable is the fast variable, called the active variable, which represents the voltage of the cell membrane. The second variable is the slow variable, which represents some time-dependent physical quantity such as the conductivity of the flow of ions across the cell membrane. The FitzHugh-Nagumo model is represented by the following system, using the notation as in Ambrosio et al. (2013, 2012):

$$\begin{cases} \varepsilon \dfrac{du}{dt} = f(u) - v, \\ \dfrac{dv}{dt} = au - bv + c, \end{cases} \tag{1}$$

where a, b and c are constant (a and b are positive), t presents the time, $0 < \varepsilon < 1$ and $f(u) = -u^3 + 3u$.

However, this system is not strong enough to reflect the propagation of cell voltage in space (along the cell body), so the cable equations are used here by adding the Laplace operator to the system (1) as follows:

$$\begin{cases} \varepsilon \dfrac{\partial u}{\partial t} = \varepsilon u_t = f(u) - v + d\Delta u, \\ \dfrac{\partial v}{\partial t} = v_t = au - bv + c, \end{cases} \tag{2}$$

where $u = u(x,t),\ v = v(x,t),\ (x,t) \in \Omega \times \mathbb{R}^+;$ d is positive constant; Δu is Laplace operator of u; $\Omega \subset \mathbb{R}^N$ is a uniformly bounded open set and the system satisfies the Neumann zero flux condition on the boundary (Ermentrout, 2009); N is a positive integer. This system consists of two parabolic nonlinear partial differential equations, allowing a wide variety of physiologically voltage-related and diverse shapes of cell membranes to be expressed (Corson, 2009; Ambrosio et al., 2013; Ermentrout, 2009). Note that the first equation, also known as the cable equation (Ermentrout, 2009), describes the flow of potentials along the body of a cell (Izhikevich, 2007; Ermentrout, 2009).

The system (2) is considered as a model of a cell from which a neural network consisting of n systems (2) is linked together by the following system:

$$\begin{cases} \varepsilon u_{it} = f(u_i) - v_i + d\Delta u_i - h(u_i, u_j), \\ v_{it} = au_i - bv_i + c, \end{cases} \qquad i, j = 1, ..., n, i \neq j, \tag{3}$$

where $(u_i, v_i), i = 1, 2, ..., n$ is defined as the system (2).

Function h is the coupling function describing the type of association between cells ith and jth. There are two types of connections between cells: chemical and electrical. This study only focuses on electrical connection, then the coupling function is linear and is given by the following formula:

$$h(u_i, u_j) = g_n \sum_{j=1}^{n} c_{ij}(u_i - u_j), \quad i = 1, 2, ..., n, \tag{4}$$

where g_n is a constant describing the coupling strength (Belykh et al., 2005; Corson, 2009). The coefficients c_{ij} are the elements of the connectivity matrix $C_n = (c_{ij})_{n \times n}$ defined by: $c_{ij} = 1$ if u_i and u_j are coupled, $c_{\text{ij}} = 0$ if u_i and u_j are not coupled, where $i, j = 1, 2, ..., n, i \neq j$.

In recent years, synchronization has been widely studied in many fields, many natural phenomena also reflect synchronization such as the movement of birds forming clouds, the movement of schools of carp in a lake, the movement of a parade, the reception and transmission of information by a group of cells, etc. (Keener et al., 2009; Murray, 2010; Nagumo et al., 1962). Therefore, research on synchronization in networks of cells is extremely necessary. In this article, we present numerical results comparing the speed of synchronization on ring networks of FHN cells with unidirectional and bidirectional coupling.

2 Mathematical definition for synchronization

A cellular network is a physiologically interconnected system of cells. The exchange between them is mainly based on electrochemical processes, each element (node) of the network is a cell modeled by a system of reaction-diffusion equations of FHN and each edge represents a cell junction simulated by the coupling function (Aziz-Alaoui, 2006; Ambrosio et al., 2013, 2012). To make the study easy, this paper only considers the connection between cells in electrical type, i.e., the cells are linked linearly with each other. Because of the interconnectedness of a network of many cells, there will be a moment when synchronization will occur with some corresponding conditions. To make it easier to imagine, synchronization for a network of two systems means that one system will copy the properties of the other from a certain time (Definition 1). The synchronization of the network of FHN in this paper is performed on ring networks with bidirectional and unidirectional coupling.

Figure 1(a) below is an example of a ring network of 12 FHN cells with unidirectional coupling to be shown by arrows, i.e., the ith cell will connect to the $(i+1)$th cell, $(i = 1, 2, ..., 11)$, and the twelfth cell will connect to the first cell. Specifically, this kind of network can be described by the system (5):

$$\begin{cases} \varepsilon u_{it} = f(u_i) - v_i + d\Delta u_i - g_n(u_i - u_{i+1}), \\ v_{it} = au_i - bv_i + c, \\ \varepsilon u_{nt} = f(u_n) - v_n + d\Delta u_n - g_n(u_n - u_1), \\ v_{nt} = au_n - bv_n + c, \end{cases} \qquad i = 1, 2, ..., n-1. \tag{5}$$

Figure 1(b) is an example of a ring network of 12 FHN cells with bidirectional coupling, i.e., the first cell will connect to the second and twelfth cells, and the ith cell will connect to the $(i+1)$th cell and the $(i-1)$th cell, $(i = 2, 3, ..., 12)$. Finally, the twelfth cell will connect to the first and eleventh cells. Specifically, this kind of network can be described by the system (6).

$$\begin{cases} \varepsilon u_{1t} = f(u_1) - v_1 + d\Delta u_1 - g_n(u_1 - u_2) - g_n(u_1 - u_n), \\ v_{1t} = au_1 - bv_1 + c, \\ \varepsilon u_{it} = f(u_i) - v_i + d\Delta u_i - g_n(u_i - u_{i+1}) - g_n(u_i - u_{i-1}), \\ v_{it} = au_i - bv_i + c, \\ \varepsilon u_{nt} = f(u_n) - v_n + d\Delta u_n - g_n(u_n - u_1) - g_n(u_n - u_{n-1}), \\ v_{nt} = au_n - bv_n + c, \end{cases} \qquad i = 2, 3, ..., n-1. \tag{6}$$

Definition 1. Let $S_i = (u_i, v_i),\ i = 1, 2, ..., n$ and $S = (S_1, S_2, ..., S_n)$ be a network. We say that S is identically synchronous if

$$\lim_{t \to +\infty} \sum_{i=1}^{n-1} \Big(\|u_i - u_{i+1}\|_{L^2(\Omega)} + \|v_i - v_{i+1}\|_{L^2(\Omega)} \Big) = 0,$$

where $L^2(\Omega)$ is function space on Ω defined using a natural generalization of the 2-norm for finite-dimensional vector spaces.

3 Results and discussion

In this section, the results of the article are carried out using numerical methods for equation system (5) and equation system (6), in which the value of n changes from 3 to 20, $f(u) = -u^3 + 3u$, $a = 1, b = 0.001, c = 0, \varepsilon = 0.1$, $d = 0.05$.

This numerical method is implemented on C++, with $[0; T] \times \Omega = [0; 200] \times [0; 100] \times [0; 100]$.

By using numerical methods, the research results allow us to find the small enough coupling strength needed for synchronization to occur in a network of cells. Specifically, the results

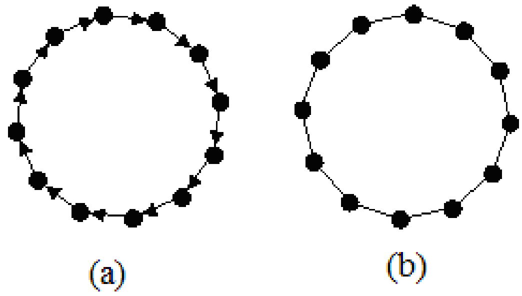

Figure 1: Two ring networks of 12 linearly interconnected cells. Each node is understood as a neuron presented by a system of reaction-diffusion equations of FHN type, and each edge represents the connection between cells. Unidirectional coupling is represented by arrows (a), while bidirectional coupling does not have arrows (b).

shown in Figure 2 describe the synchronization phenomenon in the ring network of 3 systems of reaction-diffusion equations of FHN type with unidirectional coupling. The result shows that the synchronization is performed since the value $g_3 = 0.04$. Figures 2(a), 2(b), 2(f), 2(g), 2(k), 2(l), 2(p), 2(q) represent the synchronization errors of the coupled solutions:

$$(u_1(x_1, x_2, t), u_2(x_1, x_2, t))$$

and

$$(u_2(x_1, x_2, t), u_3(x_1, x_2, t)),$$

where $t \in [0; T]$ and for all $(x_1, x_2) \in \Omega$. In Figure 2(p) and 2(q) with $g_3 = 0.04$, the simulation shows that the synchronization errors reach zero which means, for t larger enough:

$$u_1(x_1, x_2, t) \approx u_2(x_1, x_2, t)$$

and

$$u_2(x_1, x_2, t) \approx u_3(x_1, x_2, t),$$

for all $(x_1, x_2) \in \Omega$.

Figure 2(c), 2(d), 2(e), 2(h), 2(i), 2(j), 2(m), 2(n), 2(o), 2(r), 2(s), 2(t) represent the solutions $u_i(x_1, x_2, 190), i = 1, 2, 3$, of the network from when no synchronization has occurred until they have the same shape, i.e, the synchronization is performed.

Before synchronization with $g_3 = 0.01$, Figure 2(a) represents the synchronization error between u_2 and u_1, for all $(x_1, x_2) \in \Omega$; Figure 2(b) represents the synchronization error between u_3 and u_2; Figure 2(c) represents a solution $u_1(x_1, x_2, 190)$; similarly, Figure 2(d) and 2(e) represent the solutions $u_2(x_1, x_2, 190)$ and $u_3(x_1, x_2, 190)$ when they are coupled together; the results are similarly done for $g_3 = 0.02$ (Figure 2(f), 2(g), 2(h), 2(i), 2(j)); $g_3 = 0.03$ (Figure 2(k), 2(l), 2(m), 2(n), 2(o)) and $g_3 = 0.04$ (Figure 2(p), 2(q), 2(r), 2(s), 2(t)). For $g_3 = 0.04$ the synchronization occurs, since the synchronization errors reach zero (see Figure 2(p), 2(q)).

From the above results, in the case of three linearly coupled neurons, the coupling strength over or equal to $g_3 = 0,04$, these neurons have synchronous behaviors. Similarly, by changing the value of n from 3 to 20, we have Table 1 below reporting the values of coupling strength according to the number n from 3 to 20.

Following these numerical experiments, it is easy to see that the coupling strength required to observe the synchronization in the ring networks of FHN with unidirectional coupling depends on the number of nodes in the networks. Indeed, the points in Figure 3(a) represent the values of coupling strength according to the number of nodes in the ring networks with unidirectional coupling from Table 1, and we would like to find a relationship between the number of neurons n

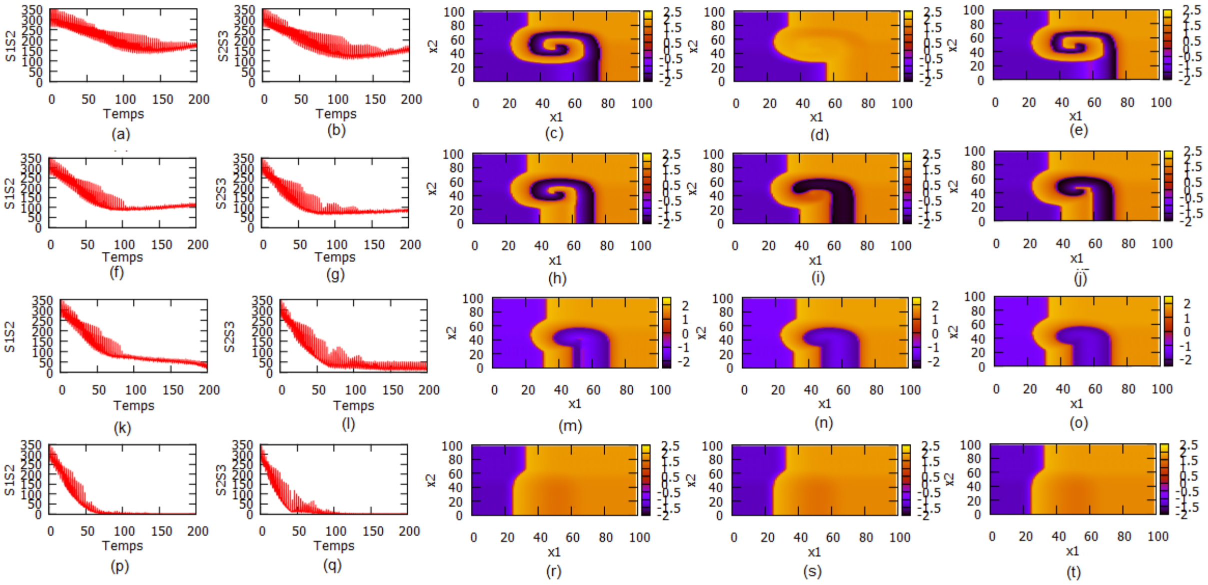

Figure 2: Synchronization in the ring network of 3 systems of FHN with unidirectional coupling.

Table 1: Minimal coupling strength necessary to observe the synchronization in the ring networks with unidirectional coupling

n		3	4	5	6
g_{syn}		0.04	0.045	0.053	0.065
n	7	8	9	10	11
g_{syn}	0.08	0.095	0.11	0.13	0.15
n		12	13	14	15
g_{syn}		0.17	0.2	0.22	0.25
n	16	17	18	19	20
g_{syn}	0.28	0.32	0.35	0.38	0.42

and the coupling strength reported in Table 1 by using interpolation method. This relationship is presented by the following function:

$$g_n = 0.00094n^2 + 0.00063n + 0.03, \tag{7}$$

where n is the number of nodes in the ring networks with unidirectional coupling. In Figure 3(a), the function (7) is represented by a curve where the points corresponding to the coupling strengths are almost on. It means that the coupling strength necessary to obtain the synchronization in the ring network with unidirectional coupling follows the law presented by (7). These simulations show that the larger the number of nodes is, the larger the coupling strength is. It means that synchronization is difficult to achieve when the value of coupling strength in the ring networks with unidirectional coupling becomes larger.

Similarly, for the ring networks of FHN with bidirectional coupling, the results in Table 2 below show the change of coupling strength corresponding to the increase in the number of nodes from 3 to 20.

In Figure 3(b), the points represent the values of coupling strength according to the number of nodes in the ring networks from Table 2, and we would like to find a function presenting the relation between the number of nodes and the coupling strength reported in Table 2 by using

Table 2: Minimal coupling strength necessary to observe the synchronization in the ring network with bidirectional coupling

n		3	4	5	6
g_{syn}		0.025	0.027	0.03	0.035
n	7	8	9	10	11
g_{syn}	0.04	0.048	0.05	0.065	0.075
n		12	13	14	15
g_{syn}		0.088	0.1	0.12	0.13
n	16	17	18	19	20
g_{syn}	0.15	0.17	0.18	0.2	0.23

interpolation method. This function is as follows:

$$g_n = 0.00065n^2 - 0.0027n + 0.027, \tag{8}$$

where n is the number of nodes in the ring network with bidirectional coupling. In Figure 3(b), the function (8) is represented by a curve where the points corresponding to the coupling strengths are almost on. It means that the coupling strength necessary to obtain the synchronization in the ring networks with bidirectional coupling follows the law presented by (8). These simulations also show that synchronization is more difficult to achieve when the number of nodes n in the ring networks with bidirectional coupling becomes greater.

In summary, based on Figure 3, it can be seen that the coupling strength required for the synchronization in the ring networks of FHN in both cases tends to increase as the number of nodes is increased, i.e., when the number of nodes in the ring networks increases, synchronization is more difficult to occur. Moreover, the comparison between the two results in Figures 3(a) and 3(b) shows that the synchronization speed of the ring networks with bidirectional coupling will be faster than the ring networks with unidirectional coupling over the same number of neurons. This is completely consistent with the law of nature when the more links and information exchanged between members in the networks, the easier it will be to have mutual synchronization.

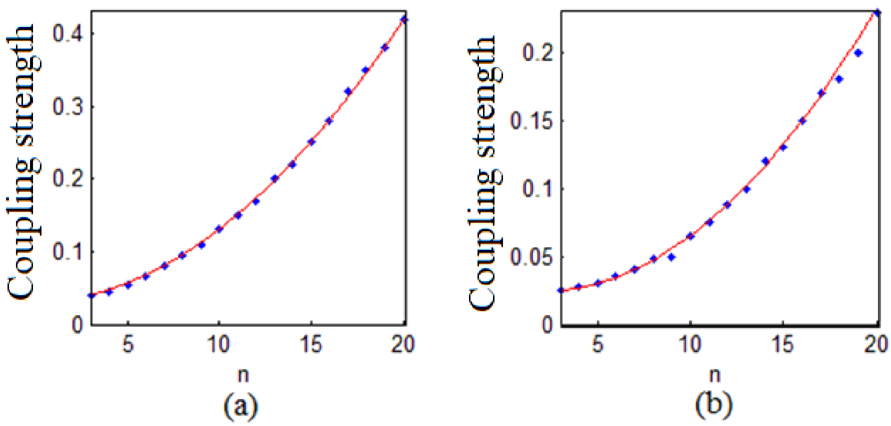

Figure 3: The evolution of the coupling strength with respect to the number of nodes in the ring networks with unidirectional coupling (a), and bidirectional coupling (b).

4 Conclusion

This article provides results on the speed of synchronization between two ring network structures of reaction-diffusion systems of the FitzHugh-Nagumo type. The results show that the coupling strength required for synchronization in both cases tends to increase as the number of nodes increases. It means if the number of nodes increases, the ring network becomes more difficult for synchronization to occur. In addition, the ring network with bidirectional coupling is more susceptible to synchronization than the network with unidirectional coupling over the same number of cells. In the next paper, the author will study the speed of synchronization of spiral solutions in the ring network with chemical connection.

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